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Oscillation criteria for second-order neutral delay differential equations

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Abstract. New sufficient conditions for oscillation of second-order neutral half-linear delay differential equations are given. Our results essentially improve, complement and simplify a number of related ones in the literature, especially those from a recent paper by [R. P. Agarwal, Ch. Zhang, T. Li, *Appl. Math. Comput.* 274(2016), 178–181]. An example illustrates the value of the results obtained.

Keywords: half-linear neutral differential equation, delay, second-order, oscillation.

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1 Introduction

The aim of this work is to study the oscillation of the second-order half-linear neutral delay differential equation

$$\left(r\left(z'\right)^{\alpha}\right)'(t)+q(t)x^{\alpha}(\sigma(t))=0, \qquad t\geq t_0>0, \tag{1.1}$$

where $z(t)=x(t)+p(t)x(\tau(t))$. Throughout, we assume that

(H_1) $\alpha>0$ is a quotient of odd positive integers;

(H_2) $r\in\mathcal{C}([t_0,\infty),(0,\infty))$ satisfies

$$\pi(t_0):=\int_{t_0}^{\infty}r^{-1/\alpha}(s)\mathrm{d}s<\infty;$$

(H_3) the delay functions σ, $\tau\in\mathcal{C}^1([t_0,\infty),\mathbb{R})$ satisfy $\tau(t)$, $\sigma(t)\leq t$, $\sigma'(t)>0$ and $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty$;

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(H$_4$) q, $p \in \mathcal{C}([t_0,\infty), [0,\infty))$, $0 \le p(t) < 1$ and q does not vanish identically on any half-line of the form $[t_*,\infty)$, $t_* \ge t_0$;

(H$_5$) $p(t) < \frac{\pi(t)}{\pi(\tau(t))}$.

Under a solution of equation (1.1), we mean a function $x \in \mathcal{C}([t_a,\infty),\mathbb{R})$ with $t_a = \min\{\tau(t_b),\sigma(t_b)\}$, for some $t_b \ge t_0$, which has the property $r\,(z')^\alpha \in \mathcal{C}^1([t_a,\infty),\mathbb{R})$ and satisfies (1.1) on $[t_b,\infty)$. We only consider those solutions of (1.1) which exist on some half-line $[t_b,\infty)$ and satisfy the condition

$$\sup\{|x(t)| : t_c \le t < \infty\} > 0 \quad \text{for any} \;\; t_c \ge t_b.$$

As is customary, a solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. The equation itself is termed oscillatory if all its solutions oscillate.

The problem of determining oscillation criteria for particular functional differential equations has been a very active research area in the past decades, and many references and summaries of known results can be found in the monographs by Agarwal et al. [1–3] and Győri and Ladas [7].

In a neutral delay differential equation, the highest-order derivative of the unknown function appears both with and without delay. The qualitative study of such equations has, besides its theoretical interest, significant practical importance. This is due to fact that neutral differential equations arise in various phenomena including problems concerning electric networks containing lossless transmission lines (as in high speed computers where such lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar, and in the solution of variational problems with time delays. We refer the reader to Hale's monograph [8] for further applications in science and technology.

In fact, the assumption

$$\pi(t_0) = \infty$$

has been commonly used in the literature in order to ensure that any possible nonoscillatory, say positive solution, x of (1.1) satisfies

$$x(t) \ge (1 - p(t))z(t). \tag{1.2}$$

There is, however, much current interest in the study of oscillation of (1.1) in the case when (H_2) holds, and consequently, the inequality (1.2) does not hold generally.

In particular, Xu and Meng [17] and Mařík [14] gave conditions under which (1.1) is either oscillatory or the solution approaches zero eventually. Ye and Xu [18] established further results ensuring that every solution of (1.1) is oscillatory. Unfortunately, as discussed in [9], some inaccuracies in their proofs prevented the successful application of the results obtained. Therefore, Han et al. [9] continued the work on this subject to obtain new oscillation criteria for (1.1), which we present below for convenience of the reader.

Theorem A (See [9, Theorem 2.1])**.** *Assume* (H_1)–(H_4) *and*

$$p'(t) \le 0 \quad \textit{and} \quad \sigma(t) \le \tau(t) = t - \tau_0 \quad \textit{for } t \ge t_0. \tag{1.3}$$

If there exists a function $\rho \in \mathcal{C}^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\rho(s)q(s)\,(1 - p(\sigma(s)))^\alpha - \frac{\left((\rho'(s))_+\right)^{\alpha+1} r(\tau(s))}{(\alpha+1)^{\alpha+1}\rho^\alpha(s)(\tau'(s))^\alpha} \right) \mathrm{d}s = \infty \tag{1.4}$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(\frac{q(s)\pi^{\alpha}(s)}{(1+p(s))^{\alpha}}-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\pi(s)r^{1/\alpha}(s)}\right)=\infty, \tag{1.5}$$

then (1.1) *is oscillatory.*

Theorem B (See [9, Theorem 2.2]). *Assume* (H_1)–(H_4) *and* (1.3). *If there exists a function* $\rho \in \mathcal{C}^1([t_0,\infty),(0,\infty))$ *such that* (1.4) *holds and for all* $t_1 \geq t_0$*,*

$$\limsup_{t\to\infty}\int_{t_1}^{t} r^{-1/\alpha}(v)\left(\int_{t_1}^{v} q(u)\left(\frac{1}{1+p(u)}\right)^{\alpha}(\pi(u))^{\alpha}\,\mathrm{d}u\right)^{1/\alpha}\mathrm{d}v=\infty, \tag{1.6}$$

then (1.1) *is oscillatory.*

Similar results to those above have been obtained in [11,13]. Using the generalized Riccati substitution, Agarwal et al. [4] have recently proved less-restrictive oscillation criteria for (1.1) without requiring condition (1.3).

Theorem C (See [4, Theorem 2.2]). *Assume* (H_1)–(H_5)*,* $\alpha \geq 1$*, and there exist functions* ρ*,* $\delta \in \mathcal{C}^1([t_0,\infty),(0,\infty))$ *such that* (1.4) *holds and*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(\psi(s)-\frac{\delta(s)r(s)\left((\varphi(s))_{+}\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right)\mathrm{d}s=\infty, \tag{1.7}$$

where

$$\psi(t):=\delta(t)\left(q(t)\left(1-p(\sigma(t))\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\right)^{\alpha}+\frac{1-\alpha}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)}\right),$$

$$\varphi(t):=\frac{\delta'(t)}{\delta(t)}+\frac{1+\alpha}{r^{1/\alpha}(t)\pi(t)},\qquad (\varphi(t))_{+}=\max\{\varphi(t),0\}.$$

Then (1.1) *is oscillatory.*

Very recently, Džurina and Jadlovská [6] established, contrary to most existing results, one-condition oscillation criteria for a special case of (1.1), namely,

$$\left(r\left(x'\right)^{\alpha}\right)'(t)+q(t)x^{\alpha}(\sigma(t))=0. \tag{1.8}$$

Theorem D (See [6, Theorem 2]). *Assume* (H_1)–(H_4)*. If*

$$\int^{\infty}\left(\frac{1}{r(t)}\int^{t}q(s)\pi^{\alpha}(\sigma(s))\mathrm{d}s\right)^{1/\alpha}\mathrm{d}t=\infty, \tag{1.9}$$

then (1.8) *is oscillatory.*

Theorem E (See [6, Theorem 3]). *Assume* (H_1)–(H_4)*. If, for all* $t_1 \geq t_0$ *large enough,*

$$\limsup_{t\to\infty}\pi^{\alpha}(t)\int_{t_1}^{t}q(s)\mathrm{d}s>1, \tag{1.10}$$

then (1.8) *is oscillatory.*

One purpose of this paper is to further improve, complement, and simplify Theorems A–C. The organization is as follows. Firstly, we extend Theorems D and E to be applicable on (1.1). The newly obtained couple of criteria ensure oscillation of (1.1) without verifying the extra condition (1.4), which has been (or its similar form) traditionally imposed in all results reported in the literature (see [4, 9, 11–14, 16–18, 20]).

Secondly, we present a comparison result in which the oscillation of (1.1) is deduced from that of a first-order delay differential equation. If, however, this criterion does not apply, we are able to obtain lower bounds of solutions to (1.1) in order to achieve a qualitatively stronger result in case of $\sigma(t) < t$.

Thirdly, following Agarwal et al. [4], we introduce a generalized Riccati substitution

$$w := \delta\left(\frac{r\,(z')^{\alpha}}{z^{\alpha}} + \frac{1}{\pi^{\alpha}}\right). \tag{1.11}$$

By careful observation and employing some inequalities of different type, we provide a criterion which is equally sharp as that in [4, Theorem 1] for Euler-type differential equations with $\sigma(t) = t$ (see Example 2.11), but

(a) applies for any $\alpha > 0$,

(b) has a significantly simpler form compared to (1.7),

(c) essentially takes into account the influence of delay argument $\sigma(t)$, which has been neglected in all previous results,

(d) in view of the technique used is in a nontraditional form ($\limsup \cdot > 1$ instead of $\limsup \cdot = \infty$) and thus can be applied to different equations which cannot be covered by the above-mentioned known results.

Moreover, as can be seen from Corollaries 2.8–2.10, this result improves Theorems A and C also for the nonneutral case, i.e., when $p(t) = 0$.

2 Main results

In what follows, all occurring functional inequalities are assumed to hold eventually, that is, they are satisfied for all t large enough. As usual and without loss of generality, we can deal only with eventually positive solutions of (1.1).

Let us define

$$Q(t) := q(t)\left(1 - p(\sigma(t))\frac{\pi\left(\tau(\sigma(t))\right)}{\pi(\sigma(t))}\right)^{\alpha}, \quad \tilde{Q}(t) = \left(\frac{1}{r(t)}\int_{t_1}^{t} Q(s)\mathrm{d}s\right)^{1/\alpha},$$

where $t_1 \in [t_0, \infty)$. By assumption (H_5), we note that the function Q is positive.

Theorem 2.1. *Assume* (H_1)–(H_5). *If*

$$\int^{\infty}\left(\frac{1}{r(t)}\int^{t} Q(s)\pi^{\alpha}(\sigma(s))\mathrm{d}s\right)^{1/\alpha}\mathrm{d}t = \infty \tag{2.1}$$

then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that x is a positive solution of (1.1) on $[t_0, \infty)$. Then there exists $t_1 \geq t_0$ such that $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Obviously, for all $t \geq t_1$, $z(t) \geq x(t) > 0$ and $r(t)\left(z'(t)\right)^\alpha$ is nonincreasing since

$$\left(r\left(z'\right)^\alpha\right)'(t) = -q(t)x^\alpha(\sigma(t)) \leq 0. \tag{2.2}$$

Therefore, z' is either eventually negative or eventually positive. We will consider each case separately.

Assume first that $z' < 0$ on $[t_1, \infty)$. Since

$$z(t) \geq -\int_t^\infty r^{-1/\alpha}(s)r^{1/\alpha}(s)z'(s)\mathrm{d}s \geq -\pi(t)r^{1/\alpha}(t)z'(t), \tag{2.3}$$

it follows that

$$\left(\frac{z}{\pi}\right)'(t) \geq 0.$$

In view of the definition of z, we get

$$x(t) = z(t) - p(t)x(\tau(t)) \geq z(t) - p(t)z(\tau(t)) \geq z(t)\left(1 - p(t)\frac{\pi(\tau(t))}{\pi(t)}\right),$$

and consequently, (2.2) becomes

$$\begin{aligned}\left(r\left(z'\right)^\alpha\right)'(t) &\leq -q(t)\left(1 - p(\sigma(t))\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\right)^\alpha z^\alpha(\sigma(t)) \\ &= -Q(t)z^\alpha(\sigma(t)).\end{aligned} \tag{2.4}$$

Taking into account the monotonicity of $r(t)\left(z'(t)\right)^\alpha$, we have

$$-r(t)\left(z'(t)\right)^\alpha \geq -r(t_1)\left(z'(t_1)\right)^\alpha =: \gamma > 0 \quad \text{for all } t \geq t_1,$$

which in view of (2.3) implies

$$z(t) \geq \gamma^{1/\alpha}\pi(t) \quad \text{for all } t \geq t_1. \tag{2.5}$$

Combining (2.4) with (2.5) yields the inequality

$$\left(r\left(z'\right)^\alpha\right)'(t) \leq -\gamma Q(t)\pi^\alpha(\sigma(t)) \quad \text{for all } t \geq t_1. \tag{2.6}$$

Integrating (2.6) from t_1 to t, we obtain

$$r(t)\left(z'(t)\right)^\alpha \leq r(t_1)\left(z'(t_1)\right)^\alpha - \gamma\int_{t_1}^t Q(s)\pi^\alpha(\sigma(s))\mathrm{d}s \leq -\gamma\int_{t_1}^t Q(s)\pi^\alpha(\sigma(s))\mathrm{d}s. \tag{2.7}$$

Integrating (2.7) from t_1 to t and taking (2.1) into account yield

$$z(t) \leq z(t_1) - \gamma^{1/\alpha}\int_{t_1}^t \left(\frac{1}{r(s)}\int_{t_1}^s Q(u)\pi^\alpha(\sigma(u))\mathrm{d}u\right)^{1/\alpha}\mathrm{d}s \to -\infty \quad \text{as } t \to \infty,$$

a contradiction.

Assume now that $z' > 0$ on $[t_1, \infty)$. Then $x(t) \geq (1 - p(t))z(t)$ and (2.2) becomes

$$\left(r\left(z'\right)^\alpha\right)'(t) \leq -q(t)(1 - p(\sigma(t)))^\alpha z^\alpha(\sigma(t)). \tag{2.8}$$

Since $\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))} \geq 1$, we have

$$1 - p(\sigma(t)) \geq 1 - p(\sigma(t)) \frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))} \tag{2.9}$$

eventually, say for $t \geq t_2$, $t_2 \in [t_1, \infty)$. On the other hand, it follows from (2.1) and (H_2) that $\int_{t_1}^{t} Q(s)\pi^{\alpha}(\sigma(s))\mathrm{d}s$ must be unbounded. Further, since $\pi'(t) < 0$, it is easy to see that

$$\int_{t_1}^{t} Q(s)\mathrm{d}s \to \infty \quad \text{as } t \to \infty. \tag{2.10}$$

Integrating (2.8) from t_2 to t and using (2.9) in the resulting inequality, we get

$$\begin{aligned} r(t)\left(z'(t)\right)^{\alpha} &= r(t_2)\left(z'(t_2)\right)^{\alpha} - \int_{t_2}^{t} q(s)(1 - p(\sigma(s)))^{\alpha} z^{\alpha}(\sigma(s))\mathrm{d}s \\ &\leq r(t_2)\left(z'(t_2)\right)^{\alpha} - z^{\alpha}(\sigma(t_2)) \int_{t_2}^{t} q(s)(1 - p(\sigma(s)))^{\alpha}\mathrm{d}s \\ &\leq r(t_2)\left(z'(t_2)\right)^{\alpha} - z^{\alpha}(\sigma(t_2)) \int_{t_2}^{t} Q(s)\mathrm{d}s, \end{aligned} \tag{2.11}$$

which in view of (2.10) contradicts to the positivity of $z'(t)$ as $t \to \infty$. The proof is complete. □

Theorem 2.2. *Assume* (H_1)–(H_5). *If, for all $t_1 \geq t_0$ large enough,*

$$\limsup_{t \to \infty} \pi^{\alpha}(t) \int_{t_1}^{t} Q(s)\mathrm{d}s > 1, \tag{2.12}$$

then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that x is a positive solution of (1.1) on $[t_0, \infty)$. Then there exists $t_1 \geq t_0$ such that $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$. As in the proof of Theorem 2.1, z' is of one sign eventually.

Assume first that $z' < 0$ on $[t_1, \infty)$. Integrating (2.4) from t_1 to t, we get

$$r(t)\left(z'(t)\right)^{\alpha} \leq r(t_1)\left(z'(t_1)\right)^{\alpha} - \int_{t_1}^{t} Q(s) z^{\alpha}(\sigma(s))\mathrm{d}s \leq -z^{\alpha}(\sigma(t)) \int_{t_1}^{t} Q(s)\mathrm{d}s. \tag{2.13}$$

Using that (2.3) holds and $z(\sigma(t)) \geq z(t)$ in (2.13), we obtain

$$-r(t)\left(z'(t)\right)^{\alpha} \geq -r(t)\left(z'(t)\right)^{\alpha} \pi^{\alpha}(t) \int_{t_1}^{t} Q(s)\mathrm{d}s. \tag{2.14}$$

Cancelling $-r(t)\left(z'(t)\right)^{\alpha}$ on both sides of (2.14) and taking the lim sup on both sides of the resulting inequality, we arrive at a contradiction with (2.12).

Assume that $z' > 0$ on $[t_1, \infty)$. Except the fact that (2.10) follows now from (2.12) and (H_2), this part of proof is similar to that of Theorem 2.1 and so we omit it. □

Remark 2.3. When $p(t) \equiv 0$, conditions (2.1) and (2.12) reduce to (1.9) and (1.10), respectively.

Next, we give the following oscillation result which is applicable for the delay case only, i.e., when $\sigma(t) < t$.

Theorem 2.4. *Assume* (H_1)–(H_5). *If*

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t}\tilde{Q}(s)\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{2.15}$$

then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that x is a positive solution of (1.1) on $[t_0,\infty)$. Then there exists $t_1 \geq t_0$ such that $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$. As in the proof of Theorem 2.1, z' is of one sign eventually.

Assume first that $z' < 0$ on $[t_1,\infty)$. From (2.13), it is easy to see that z is a solution of the first-order delay differential inequality

$$z'(t) + \tilde{Q}(t)z(\sigma(t)) \leq 0. \tag{2.16}$$

In view of [15, Theorem 1], the associated delay differential equation

$$z'(t) + \tilde{Q}(t)z(\sigma(t)) = 0 \tag{2.17}$$

also has a positive solution. However, it is well-known (see, e.g., [10, Theorem 2]) that condition (2.15) implies oscillation of (2.17). This in turn means that (1.1) cannot have a positive solution, a contradiction.

Assume that $z' > 0$ on $[t_1,\infty)$. If suffices to note that

$$\int_{t_0}^{\infty}\tilde{Q}(s)\mathrm{d}s = \infty \tag{2.18}$$

is necessary for the validity of (2.15). Then, except the fact that (2.10) follows now from (2.18) and (H_2), this part of proof is similar to that of Theorem 2.1 and so we omit it. The proof is complete. □

It is obvious that if

$$\int_{\sigma(t)}^{t}\tilde{Q}(s)\mathrm{d}s \leq \frac{1}{\mathrm{e}}, \tag{2.19}$$

then Theorem 2.4 does not apply. If, however, (2.19) holds and z is a positive solution of (2.16), then it is possible to obtain lower bounds of $\frac{z(\sigma(t))}{z(t)}$ which will play an important role in proving the next theorem. Zhang and Zhou [19] obtained such bounds for (2.17) by defining a sequence $\{f_n(\rho)\}$ by

$$f_0(\rho) = 1, \qquad f_{n+1}(\rho) = \mathrm{e}^{\rho f_n(\rho)}, \qquad n \in \mathbb{N}_0, \tag{2.20}$$

where ρ is a positive constant satisfying

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t}\tilde{Q}(s)\mathrm{d}s \geq \rho \quad \text{for } t \geq t_1. \tag{2.21}$$

They showed that, for $\rho \in (0,1/\mathrm{e}]$, the sequence is increasing and bounded above and $\lim_{t\to\infty} f_n(\rho) = f(\rho) \in [1,\mathrm{e}]$, where $f(\rho)$ is a real root of the equation

$$f(\rho) = \mathrm{e}^{\rho f(\rho)}. \tag{2.22}$$

We essentially use their result in the following lemma.

Lemma 2.5. *Assume that* (2.21) *holds for* $\rho > 0$ *and let* x *be a positive solution of* (1.1) *with* $z > 0$ *satisfying* $z' < 0$ *on* $[t_1, \infty)$. *Then there exists* $t_2 \geq \sigma^{-(2+n)}(t_1)$ *such that, for some* $n \in \mathbb{N}_0$,

$$\frac{z(\sigma(t))}{z(t)} \geq f_n(\rho) \quad \text{for } t \geq t_2, \tag{2.23}$$

where $f_n(\rho)$ *is defined by* (2.20).

Proof. Let x be a positive solution of (1.1) with $z > 0$ satisfying $z' < 0$ on $[t_1, \infty)$. Then as in the proof of Theorem 2.4, one can obtain that z is a positive solution of the first-order delay differential inequality (2.16). Proceeding in the same manner as in the proof of [19, Lemma 1], we see that the estimate (2.23) holds. □

Finally, we recall another auxiliary result which is extracted from Erbe et al. [16, Lemma 2.3].

Lemma 2.6. *Let* $g(u) = Au - B(u-C)^{\frac{\alpha+1}{\alpha}}$, *where* $B > 0$, A *and* C *are constants,* α *is a quotient of odd positive numbers. Then* g *attains its maximum value on* $\mathbb{R}$ *at* $u^* = C + \left(\frac{\alpha}{\alpha+1}\frac{A}{B}\right)^{\alpha}$ *and*

$$\max_{u \in \mathbb{R}} g(u) = g(u^*) = AC + \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{A^{\alpha+1}}{B^\alpha}. \tag{2.24}$$

Let us define the sequence of functions $\{\psi_n(t)\}$ by

$$\psi_n(t) := \begin{cases} Q(t), & \text{if } \sigma(t) = t, \\ f_n^\alpha(\rho)\, Q(t), & \text{if } \sigma(t) < t, \end{cases}$$

where $n \in \mathbb{N}_0$, $\rho \in (0, 1/\mathrm{e}]$ satisfies (2.21) and $f_n(\rho)$ is defined by (2.20).

Theorem 2.7. *Assume* (H_1)–(H_5). *If there exist functions* ρ, $\delta \in C^1([t_0, \infty), (0, \infty))$ *and* $T \in [t_0, \infty)$ *such that* (1.4) *holds and, for some* $n \in \mathbb{N}_0$,

$$\limsup_{t \to \infty} \left\{ \frac{\pi^\alpha(t)}{\delta(t)} \int_T^t \left(\delta(s)\psi_n(s) - \frac{r(s)\,(\delta'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\delta^\alpha(s)} \right) \mathrm{d}s \right\} > 1, \tag{2.25}$$

then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that x is a positive solution of (1.1) on $[t_0, \infty)$. Then there exists $t_1 \geq t_0$ such that $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$. As in the proof of Theorem 2.1, we have that z' is of one sign eventually.

Assume first that $z' < 0$ on $[t_1, \infty)$. Proceeding as in the proof of Theorem 2.1, we obtain that z is a solution of the inequality (2.4). Let us define the Riccati function w by (1.11), that is,

$$w := \delta \left(\frac{r\,(z')^\alpha}{z^\alpha} + \frac{1}{\pi^\alpha} \right) \quad \text{on } [t_1, \infty). \tag{2.26}$$

In view of (2.3), we see that $w \geq 0$ on $[t_1, \infty)$. Differentiating (2.26), we arrive at

$$\begin{aligned} w' &= \frac{\delta'}{\delta} w + \delta \frac{\left(r\,(z')^\alpha\right)'}{z^\alpha} - \alpha\delta r \left(\frac{z'}{z}\right)^{\alpha+1} + \frac{\alpha\delta}{r^{1/\alpha}\pi^{\alpha+1}} \\ &\leq \frac{\delta'}{\delta} w + \delta \frac{\left(r\,(z')^\alpha\right)'}{z^\alpha} - \frac{\alpha}{(\delta r)^{1/\alpha}} \left(w - \frac{\delta}{\pi^\alpha} \right)^{(\alpha+1)/\alpha} + \frac{\alpha\delta}{r^{1/\alpha}\pi^{\alpha+1}}. \end{aligned} \tag{2.27}$$

Combining (2.23) from Lemma 2.5 with (2.4), we have

$$\left(r\left(z'\right)^{\alpha}\right)' \le -\psi_n z^{\alpha} \tag{2.28}$$

for some $n \in \mathbb{N}_0$ on $[t_2, \infty)$, where $t_2 \in [\sigma^{-(2+n)}(t_1), \infty)$. It follows from (2.27) that

$$w' \le -\delta(t)\psi_n + \frac{\delta'}{\delta}w - \frac{\alpha}{(\delta r)^{1/\alpha}}\left(w - \frac{\delta}{\pi^{\alpha}}\right)^{(\alpha+1)/\alpha} + \frac{\alpha\delta}{r^{1/\alpha}\pi^{\alpha+1}}.$$

We use (2.24) with

$$A := \frac{\delta'}{\delta}, \qquad B := \frac{\alpha}{(\delta r)^{1/\alpha}}, \qquad C := \frac{\delta}{\pi^{\alpha}}$$

to obtain

$$\begin{aligned} w' &\le -\delta\psi_n + \frac{\delta'}{\pi^{\alpha}} + \frac{r\left(\delta'\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\delta^{\alpha}} + \frac{\alpha\delta}{r^{1/\alpha}\pi^{\alpha+1}} \\ &= -\delta\psi_n + \left(\frac{\delta}{\pi^{\alpha}}\right)' + \frac{r\left(\delta'\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\delta^{\alpha}}. \end{aligned} \tag{2.29}$$

Integrating (2.29) from t_2 to t, we arrive at

$$\int_{t_2}^{t}\left(\delta(s)\psi_n(s) - \frac{r(s)\left(\delta'(s)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\delta^{\alpha}(s)}\right)\mathrm{d}s - \frac{\delta(t)}{\pi^{\alpha}(t)} + \frac{\delta(t_2)}{\pi^{\alpha}(t_2)} \le w(t_2) - w(t).$$

In view of the definition of w, we are led to

$$\int_{t_2}^{t}\left(\delta(s)\psi_n(s) - \frac{r(s)\left(\delta'(s)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\delta^{\alpha}(s)}\right)\mathrm{d}s \le \delta(t_2)\frac{r(t_2)\left(z'(t_2)\right)^{\alpha}}{z^{\alpha}(t_2)} - \delta(t)\frac{r(t)\left(z'(t)\right)^{\alpha}}{z^{\alpha}(t)}. \tag{2.30}$$

On the other hand, it follows from (2.3) that

$$-\frac{\delta(t)}{\pi^{\alpha}(t)} \le \delta(t)\frac{r(t)\left(z'(t)\right)^{\alpha}}{z^{\alpha}(t)} \le 0.$$

After substituting the above estimate into (2.30), we obtain

$$\int_{t_2}^{t}\left(\delta(s)\psi_n(s) - \frac{r(s)\left(\delta'(s)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\delta^{\alpha}(s)}\right)\mathrm{d}s \le \frac{\delta(t)}{\pi^{\alpha}(t)}. \tag{2.31}$$

Multiplying (2.31) by $\frac{\pi^{\alpha}(t)}{\delta(t)}$ and taking the $\limsup$ on both sides of the resulting inequality, we arrive at contradiction with (2.9). The proof is complete.

Assume that $z' > 0$ on $[t_1, \infty)$. Then we are back to the proof of [18, Theorem 2.1] to obtain a contradiction with (1.4). The proof is complete. □

Theorem 2.7 can be used in a wide range of applications for oscillation of (1.1) depending on the appropriate choice of functions ρ and δ. Namely, by choosing

(a) $\rho(t) \equiv 1$, $\delta(t) = \pi^{\alpha}(t)$,

(b) $\rho(t) \equiv 1$, $\delta(t) = \pi(t)$,

(c) $\rho(t) = \delta(t) \equiv 1$,

respectively, we get the following results, which are new also for the nonneutral ordinary case, i.e., when $p(t) = 0$ and $\sigma(t) = t$.

Corollary 2.8. *Assume* (H_1)–(H_5). *If*

$$\int_{t_0}^{\infty} q(s)\,(1-p(\sigma(s)))^{\alpha}\,\mathrm{d}s=\infty \tag{2.32}$$

and there exist $T\in[t_0,\infty)$ *and* $n\in\mathbb{N}_0$ *such that*

$$\limsup_{t\to\infty}\int_T^t\left(\pi^{\alpha}(s)\psi_n(s)-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\pi(s)r^{1/\alpha}(s)}\right)\mathrm{d}s>1, \tag{2.33}$$

then (1.1) *is oscillatory.*

Corollary 2.9. *Assume* (H_1)–(H_5) *and* (2.32). *If there exist* $T\in[t_0,\infty)$ *and* $n\in\mathbb{N}_0$ *such that*

$$\limsup_{t\to\infty}\pi^{\alpha-1}(t)\int_T^t\left(\pi(s)\psi_n(s)-\frac{1}{(\alpha+1)^{\alpha+1}\pi^{\alpha}(s)r^{1/\alpha}(s)}\right)\mathrm{d}s>1, \tag{2.34}$$

then (1.1) *is oscillatory.*

Corollary 2.10. *Assume* (H_1)–(H_5) *and* (2.32). *If there exist* $T\in[t_0,\infty)$ *and* $n\in\mathbb{N}_0$ *such that*

$$\limsup_{t\to\infty}\pi^{\alpha}(t)\int_T^t\psi_n(s)\mathrm{d}s>1, \tag{2.35}$$

then (1.1) *is oscillatory.*

Finally, we illustrate the importance of our results on the following example.

Example 2.11. Consider the second-order neutral differential equation

$$\left(t^{\alpha+1}\left[\left(x(t)+p_0x\left(\frac{t}{2}\right)\right)'\right]^{\alpha}\right)'+q_0x^{\alpha}(\lambda t)=0,\qquad t\geq 1, \tag{2.36}$$

where $\alpha>0$ is a quotient of odd positive integers, $q_0\in(0,\infty)$, $p_0\in[0,\sqrt[\alpha]{1/2})$ and $\lambda\in(0,1]$. It is clear that assumptions (H_1)–(H_5) hold. By Theorem 2.2, we deduce that equation (2.36) is oscillatory if

$$\alpha^{\alpha}q_0\left(1-\sqrt[\alpha]{2}p_0\right)^{\alpha}>1. \tag{2.37}$$

By Theorem 2.4, the same conclusion holds for (2.36) if $\lambda<1$ and

$$q_0^{1/\alpha}\left(1-\sqrt[\alpha]{2}p_0\right)\ln\left(\frac{1}{\lambda}\right)>\frac{1}{\mathrm{e}}. \tag{2.38}$$

If, however, (2.38) does not hold, we set

$$\rho:=q_0^{1/\alpha}\left(1-\sqrt[\alpha]{2}p_0\right)\ln\left(\frac{1}{\lambda}\right).$$

Clearly, since $\rho\leq 1/\mathrm{e}$, the sequence $\{f_n\}$ defined by (2.20) has a finite limit (2.22), which can be expressed as

$$f(\rho)=\lim_{n\to\infty}f_n(\rho)=-\frac{W(-\rho)}{\rho},$$

where W standardly denotes the principal branch of the Lambert function, see [5] for details.

To apply Corollary 2.8, we first note that (2.32) is satisfied. Then (2.36) is oscillatory in delay case ($\lambda < 1$) if

$$f(\rho) q_0 \left(1 - \sqrt[\alpha]{2} p_0\right)^{\alpha} > \frac{1}{(\alpha+1)^{\alpha+1}}, \tag{2.39}$$

and in ordinary case ($\lambda = 1$) if

$$q_0 \left(1 - \sqrt[\alpha]{2} p_0\right)^{\alpha} > \frac{1}{(\alpha+1)^{\alpha+1}}. \tag{2.40}$$

Note that if $p_0 = 0$ and $\alpha = 1$, then (2.40) reduces to the condition $q_0 > 1/4$, which is sharp for oscillation of the Euler differential equation

$$\left(t^2 x'(t)\right)' + q_0 x(t) = 0.$$

In fact, Theorems A and B cannot be applied in (2.36) due to (1.3). To apply Theorem C, we must require $\alpha \geq 1$. Then (2.36) is oscillatory if

$$q_0 \left(1 - \sqrt[\alpha]{2} p_0\right)^{\alpha} > \frac{1 - (1-\alpha)(\alpha+1)^{\alpha+1}}{\alpha^{\alpha+1}(\alpha+1)^{\alpha+1}}. \tag{2.41}$$

Apparently, conditions (2.40) and (2.41) are the same for $\alpha = 1$. This confirms the fact that the influence of the delay term has been neglected in previous works.

Finally, let us consider a particular case of (2.36), namely,

$$\left(t^2 \left(x(t) + \frac{1}{4} x\left(\frac{t}{2}\right)\right)'\right)' + \frac{1}{3} x\left(\frac{t}{8}\right) = 0. \tag{2.42}$$

Obviously, (2.37), (2.38) and (2.41) fail to apply. However, it is easy to verify that (2.39) reduces to $1/3 > 1/4$, which implies that (2.42) is oscillatory.

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References

[1] R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation theory for second order linear, half-linear, superlinear and sublinear dynamic equations*, Kluwer Academic Publishers, Dordrecht, 2002. MR2091751

[2] R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation theory for second order dynamic equations*, Series in Mathematical Analysis and Applications, Vol. 5, Taylor & Francis, Ltd., London, 2003. MR1965832

[3] R. P. Agarwal, M. Bohner, W.-T. Li, *Nonoscillation and oscillation: theory for functional differential equations*, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 267, Marcel Dekker, Inc., New York, 2004. MR2084730

[4] R. P. Agarwal, Ch. Zhang, T. Li, Some remarks on oscillation of second order neutral differential equations, *Appl. Math. Comput.* **274**(2016), 178–181. MR3433126

[5] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, D. E. Knuth, On the Lambert W function, *Adv. Comput. Math.* **5**(1996), No. 4, 329–359. MR1414285

[6] J. Džurina, I. Jadlovská, A note on oscillation of second-order delay differential equations, *Appl. Math. Lett.* **69**(2017), 126–132. MR3626228

[7] I. Győri, G. Ladas, *Oscillation theory of delay differential equations*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1991. MR1168471

[8] J. K. Hale, *Functional differential equations*, Oxford Applied Mathematical Sciences, Vol. 3, Springer-Verlag New York, New York–Heidelberg, 1971. MR0466837

[9] Z. Han, T. Li, S. Sun, Y. Sun, Remarks on the paper [Appl. Math. Comput. 207 (2009) 388–396], **215**(2010), *Appl. Math. Comput.*, No. 11, 3998–4007. MR2578865

[10] Y. Kitamura, T. Kusano, Oscillation of first-order nonlinear differential equations with deviating arguments, *Proc. Amer. Math. Soc.* **78**(1980), No. 1, 64–68. MR548086

[11] T. Li, Z. Han, Ch. Zhang, S. Sun, On the oscillation of second-order Emden–Fowler neutral differential equations, *J. Appl. Math. Comput.* **37**(2011), No. 1–2, 601–610. MR2831557

[12] T. Li, Y. V. Rogovchenko, Ch. Zhang, Oscillation of second-order neutral differential equations, *Funkcial. Ekvac.* **56**(2013), No. 1, 111–120. MR3099036

[13] T. Li, Y. V. Rogovchenko, Ch. Zhang, Oscillation results for second-order nonlinear neutral differential equations, *Adv. Difference Equ.* **2013**, 2013:336, 13 pp. MR3213905

[14] R. Mařík, Remarks on the paper by Sun and Meng, Appl. Math. Comput. 174 (2006), *Appl. Math. Comput.* **248**(2014), 309–313. MR3276683

[15] Ch. G. Philos, On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays, *Arch. Math. (Basel)* **36**(1981), 168–178. MR619435

[16] H. Wu, L. Erbe, A. Peterson, Oscillation of solution to second-order half-linear delay dynamic equations on time scales, *Electron. J. Differential Equations* **2016**, No. 71, 1–15. MR3489997

[17] R. Xu, F. Meng, Some new oscillation criteria for second order quasi-linear neutral delay differential equations, *Appl. Math. Comput.* **182**(2006), No. 1, 797–803. MR2292088

[18] L. Ye, Z. Xu, Oscillation criteria for second order quasilinear neutral delay differential equations, *Appl. Math. Comput.* **207**(2009), No. 2, 388–396. MR2489110

[19] B. Zhang, Y. Zhou, The distribution of zeros of solutions of differential equations with a variable delay, *J. Math. Anal. Appl.* **256**(2001), No. 1, 216–228. MR1820077

[20] Ch. Zhang, R. P. Agarwal, M. Bohner, T. Li, Oscillation of second-order nonlinear neutral dynamic equations with noncanonical operators, *Bull. Malays. Math. Sci. Soc.* **38**(2015), No. 2, 761–778. MR3323739